Advanced Calculus Midterm Exam April 27, 2011

Note: There are 8 questions with total 126 points in this exam.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (e^x \cos y, e^x \sin y)$.

   (a) (8 points) For each $(x, y) \in \mathbb{R}^2$, show that there is an open neighborhood U of (x, y) such that f has a (local) C^1 inverse defined on $f(U)$. [You may want to check that a proper theorem is applicable here].

   Solution: Since f is smooth, and, for each $(x, y) \in \mathbb{R}^2$, we have $\det Df = \det \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}$ $= e^{2x} \neq 0$. By the Inverse Function Theorem, there is an open neighborhood U of (x, y) on which f has a (local) C^1 inverse defined on $f(U)$.

   (b) (4 points) Does f have a global inverse defined on the $\mathbb{R}^2$? Give explanation to your answer.

   Solution: Since $f(x, y + 2k\pi) = f(x, y)$ for any $(x, y) \in \mathbb{R}^2$ and any $k \in \mathbb{Z}$, f is not a one-to-one function on $\mathbb{R}^2$ and it does not have a global inverse on $\mathbb{R}^2$.

2. (10 points) Let $F(x, y, z) = (xy + 2yz - 3xz, xyz + x - y - 1)$ for $x, y, z \in \mathbb{R}$. Find the differential DF at $(x, y, z) = (1, 1, 1)$, and determine whether it is possible to represent the set $S = \{(x, y, z) \mid F(x, y, z) = (0, 0)\}$ as a smooth curve parametrized by z, i.e. whether it is possible to solve for x, y in terms of z, near the point $(x, y, z) = (1, 1, 1)$.

   Solution: Direct computation gives that $DF|_{(1,1,1)} = \begin{pmatrix} y - 3z & x + 2y & 2y - 3x \\ yz + 1 & xz - 1 & xy \end{pmatrix}_{(1,1,1)} = \begin{pmatrix} -2 & 3 & -1 \\ 2 & 0 & 1 \end{pmatrix}$.
   Since $\det \begin{pmatrix} -2 & 3 \\ 2 & 0 \end{pmatrix} = -6 \neq 0$, S can be represented as a smooth curve, parametrized by z, near the point $(x, y, z) = (1, 1, 1)$ by the Implicit Function Theorem.

3. (a) (10 points) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (x + y - z, x - y + z, x^2 + y^2 + z^2 - 2yz)$. Determine the rank of DF on $\mathbb{R}^3$ and determine whether the image set $F(\mathbb{R}^3)$ is locally a smooth surface or a smooth curve.

   Solution: Direct computation gives $DF = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2x & 2y - 2z & 2z - 2y \end{pmatrix}$.
   Since $\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$ and $\begin{pmatrix} -1 \\ 1 \\ 2z - 2y \end{pmatrix} = -\begin{pmatrix} 1 \\ -1 \\ 2y - 2z \end{pmatrix}$, DF has constant rank 2 on $\mathbb{R}^3$, and the set $F(\mathbb{R}^3)$ is locally a smooth surface.

   (b) (10 points) Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $F(x, y, z) = (xy + z, x^2y^2 + 2xyz + z^2, 2 - xy - z)$. Determine the rank of DF on $\mathbb{R}^3$ and determine whether the image set $F(\mathbb{R}^3)$ is locally a smooth surface or a smooth curve.

   Solution: Direct computation gives that $DF = \begin{pmatrix} y & x & 1 \\ 2y(xy + z) & 2x(xy + z) & 2(xy + z) \\ -y & -x & -1 \end{pmatrix}$.
   Since $\big(2y(xy + z), 2x(xy + z), 2(xy + z)\big) = 2(xy + z)\big(y, x, 1\big)$ and $\big(-y, -x, -1\big) = -\big(y, x, 1\big)$, DF has constant rank 1 on $\mathbb{R}^3$, and the set $F(\mathbb{R}^3)$ is locally a smooth curve.

4. Let $f : \mathbb{R}^3 \to \mathbb{R}^2$ be defined by $f(x, y, z) = (x + y + z, x - y - 2xz)$, so that $f(0, 0, 0) = (0, 0)$ and $Df(0, 0, 0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}$.

(a) (10 points) Show that we can solve for $(x, y) = g(z)$, i.e. solve for x, y in terms of z,

Solution: In $Df(0,0,0)$, since $\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$, we can solve for $(x, y) = g(z)$, by the Implicit Function Theorem, i.e. we can solve for x, y in terms of z.

(b) (10 points) Show that $Dg(0) = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$.

Solution: By differentiating the components of f with respect to z, we get
$f_x \frac{\partial x}{\partial z} + f_y \frac{\partial y}{\partial z} + f_z|_{\text{at } (0,0,0)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{\partial x}{\partial z} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{\partial y}{\partial z} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

which is equivalent to that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \Rightarrow Dg(0) = \begin{pmatrix} \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial z} \end{pmatrix}_{\text{at } z=0} = -\frac{1}{2} \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix}$$

5. (a) (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^{\frac{1}{3}}$. For each $c > 0$, prove that f is Lipschitz on $[c, \infty)$.

Solution: For any $x, y \in [c, \infty)$, by the Mean Value Theorem, we have
$|f(x) - f(y)| = |f'(z)(x - y)| = |\frac{1}{3z^{2/3}}(x - y)|$ for some point z lying between $x, y \in [c, \infty)$
$\Rightarrow |f(x) - f(y)| = |\frac{1}{3z^{2/3}}(x - y)| \leq \frac{1}{3c^{2/3}}|x - y|$ holds for any $x, y \in [c, \infty)$.
This proves that f is Lipschitz on $[c, \infty)$.

(b) (6 points) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Prove that f is **Not** Lipschitz on $[1, \infty)$.

Solution: For any $x, y \in [1, \infty)$, by the Mean Value Theorem, we have
$|f(x) - f(y)| = |2z(x - y)|$ for some point z lying between x and y.
By letting x and y go to ∞, we note that z will go to ∞.
Thus,there does not exist a fixed number A such that $|f(x) - f(y)| \leq A|x - y|$ holds for all $x, y \in [1, \infty)$.
Hence, f is not Lipschitz on $[1, \infty)$.

(c) (6 points) Let f, g be Lipschitz maps defined on $D \subset \mathbb{R}^p$ with ranges in $\mathbb{R}^q$. Prove that $f + g$ is Lipschitz on D.

Solution: Since f, g are Lipschitz on D, there exist constant $A, B \geq 0$ such that
$\|f(x) - f(y)\| \leq A\|x - y\|$ and $\|g(x) - g(y)\| \leq B\|x - y\|$ hold for any $x, y \in D$.
This implies that $\|(f + g)(x) - (f + g)(y)| \leq \|f(x) - f(y)\| + \|g(x) - g(y)\| \leq (A + B)\|x - y\|$ holds for any $x, y \in D$.
Hence, $f + g$ is Lipschitz on D.

(d) (6 points) Give an example of Lipschitz functions $f, g : [1, \infty) \to \mathbb{R}$ and that the product fg is **Not** Lipschitz on $[1, \infty)$.

Solution: Let $f(x) = g(x) = x$ for each $x \in [1, \infty)$. Then, f and g are Lipschitz (with LIpschitz constant 1), but $(fg)(x) = x^2$ is not Lipschitz by part (b).

6. (a) (8 points) Let $\{f_n\}$ be a sequence of functions defined by $f_n(x) = \frac{x}{n}$ for each $x \in \mathbb{R}$. Show, without using Arzelà-Ascoli's theorem, that f_n converges uniformly on any closed interval $[a, b] \subset \mathbb{R}$ while the convergence in **Not** uniform on $\mathbb{R}$.

Solution: For each $x \in \mathbb{R}$, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{n} = 0 = f(x)$,
and $\lim_{n\to\infty} \sup_{[a,b]} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{[a,b]} \frac{|x|}{n} = \lim_{n\to\infty} \max\{\frac{|a|}{n}, \frac{|b|}{n}\} = 0$.
Hence, f_n converges uniformly on any closed interval $[a, b] \subset \mathbb{R}$.

Now, since $\lim_{n\to\infty} \sup_{\mathbb{R}} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{\mathbb{R}} \frac{|x|}{n} \geq \lim_{n\to\infty} \frac{|n|}{n} = 1 \neq 0$, the convergence is not uniform on $\mathbb{R}$.

(b) (8 points) Let f_n be defined on the interval $[0,1]$ by the formula

$$f_n(x) = \begin{cases} 0, & \text{if } x \in [0, 1-\frac{1}{n}], \\ nx-(n-1), & \text{if } x \in [1-\frac{1}{n}, 1]. \end{cases}$$

Show that $\lim_{n\to\infty} f_n(x)$ exists on $[0,1]$, and, without using Arzelà-Ascoli's theorem, show that this convergence is **Not** uniform on $[0,1]$.

Solution: For each $x \in [0,1)$, we have $x \in [0, 1-\frac{1}{n})$ if $n > \frac{1}{1-x}$.
This implies that $f_n(x) = 0$ for all $n > \frac{1}{1-x}$ and for all $x \in [0,1)$.
Thus, we have $\lim_{n\to\infty} f_n(x) = 0$ for each $x \in [0,1)$.
At $x = 1$, since $f_n(1) = 1$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} f_n(1) = 1$.
Hence, we have $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & \text{if } x \in [0,1) \\ 1 & \text{if } x = 1 \end{cases}$,
and, since $\lim_{n\to\infty} \sup_{[0,1]} |f_n(x) - f(x)| = \lim_{n\to\infty} \sup_{[0,1)} \big(nx - (n-1)\big) = \lim_{n\to\infty} 1 = 1 \neq 0$,
the convergence is not uniform on $[0,1]$.

7. Let $\mathscr{F} = \{f_n(x) = \frac{x^n}{n} \mid x \in [0,1],\ n = 1, 2, \ldots\}$.

(a) (4 points) Show that $\mathscr{F}$ is uniformly bounded on $[0,1]$.

Solution: Since $|f_n(x)| \leq \frac{1}{n} \leq 1$, for each $f_n \in \mathscr{F}$ and for each $x \in [0,1]$, the set $\mathscr{F}$ is uniformly bounded on $[0,1]$.

(b) (8 points) Show, without using the arzelà-Ascoli's theorem, that $\mathscr{F}$ is uniformly equicontinuous on $[0,1]$.

Solution: For each $f_n \in \mathscr{F}$ and for each $x \in (0,1)$, since $|f_n'(x)| = |x|^{n-1} \leq 1$, by the Mean Value Theorem, we have $(*)$ $\cdots$ $|f_n(x) - f_n(y)| \leq 1 \cdot |x-y|$ for any $x, y \in [0,1]$ and for all $n \in \mathbb{N}$.
This implies that $\mathscr{F}$ is uniformly equicontinuous on $[0,1]$.
(For each $\epsilon > 0$, we choose $\delta = \epsilon$ such that if $x, y \in [0,1]$ and $|x-y| < \delta$, then the inequality $(*)$ implies that $|f_n(x) - f_n(y)| < |x-y| < \delta = \epsilon$ for each $f_n \in \mathscr{F}$.)

8. (12 points) Let $\{f_n\}$ be a sequence of continuous functions with domain $D \subset \mathbb{R}^p$ and range in $\mathbb{R}^q$ and let this sequence converge uniformly on D to a function f. Prove that f is continuous on D.

Solution: Given $\epsilon > 0$, since f_n converges uniformly to f on D, there exists an $M \in \mathbb{N}$ such that $\|f_n(x) - f(x)\| < \frac{\epsilon}{3}$ for each $x \in D$.
At any point $x \in D$, since f_M is continuous at x, there exists a $\delta > 0$ such that if $y \in D$ and $\|x - y\| < \delta$ then $\|f_M(x) - f_M(y)\| < \frac{\epsilon}{3}$.
Thus, we have
$\|f(x) - f(y)\| = \|f(x) - f_M(x) + f_M(x) - f_M(y) + f_M(y) - f(y)\|$
$\leq \|f(x) - f_M(x)\| + \|f_M(x) - f_M(y)\| + \|f_M(y) - f(y)\| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$.
This implies that f is continuous at x.
Since x is an arbitrarily chosen point from D, f is continuous on D.